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1982 J. Phys. A: Math. Gen. 15 1759

(http://iopscience.iop.org/0305-4470/15/5/037)

J. Phys. A: Math. Gen. **15** (1982) 1759–1763. Printed in Great Britain

COMMENT

Comment on an exactly soluble anisotropic percolation model

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Received 7 September 1981, resubmitted 17 November 1981

Abstract. We employ a cell-to-cell position-space renormalisation group introduced for pure percolation by Reynolds *et al* to solve exactly a directed percolation problem studied by Domany and Kinzel. We show that this model has the same correlation length exponent as one-dimensional percolation provided that the proper nonlinear scaling fields are used.

The percolation problem (Broadbent and Hammersley 1957) has been a subject of intense investigation for some time (for an introduction to percolation see reviews by Stauffer 1979 and Essam 1980). The primary reasons for this interest are that percolation models exhibit a variety of interesting critical behaviour and they have been useful models for several physical processes.

One variation of the standard percolation model, directed percolation, is a particularly interesting example of these aspects of percolation. Directed percolation has been shown to be a model for Reggeon field theory (Cardy and Sugar 1980) and Markov processes that occur in chemistry and biology (Grassberger and de la Torre 1979, Schlögl 1972). It has also been shown to be in a different universality class from pure percolation (Kinzel and Yeomans 1981, Redner 1981a, Reynolds 1981, Redner and Brown 1981) and to have angle dependent exponents (Domany and Kinzel 1981, Klein and Kinzel 1981), a point we will discuss further below.

The directed percolation model is defined as follows. Consider a lattice (for example the square lattice in two dimensions of figure 1(*a*)) on which bonds are distributed at random with a probability p. In contrast to pure bond percolation, we only allow connectivity to 'flow' in one direction on a bond. We will only allow bonds to conduct from right to left if horizontal and downwards if vertical (see figure 1(*a*)). So that in figure 1(*a*) connectivity can flow from site 1 to site 2 but not from site 2 to site 1.

A correlation length $\xi_{\parallel}(\varphi)$ can be defined by writing the probability of a site at the origin being connected to a site at a position given by $\boldsymbol{R}$ as $P(R, \varphi) \cong \exp[-|\boldsymbol{R}|/\xi_{\parallel}(\varphi)]$ in the limit of large $|\boldsymbol{R}|$. In general

$$\xi_{\parallel}(\varphi) \sim (p - p_c(\varphi))^{-\nu_{\parallel}(\varphi)}$$

where $\nu_{\parallel}(\varphi)$ and $p_c(\varphi)$ depend on the angle φ that $\boldsymbol{R}$ makes with a fixed reference axis (Domany and Kinzel 1981).

† Supported by grants from ARO, ONR and NSF.

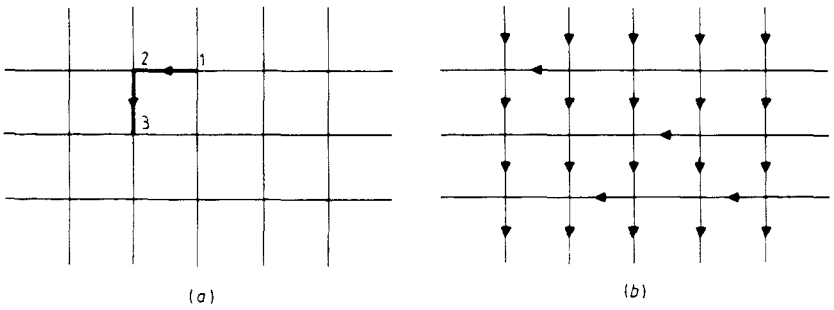

Figure 1. (*a*) A cluster of three sites in the directed percolation problem. There is a path from site 1 to site 3 but no path from site 3 to site 1. (*b*) All bonds in the vertical direction are present with probability 1.

In addition to the divergence of $\xi_{\parallel}(\varphi)$ and its angle dependence, there exists another diverging length $\xi_{\perp}(\varphi)$ defined by the probability that a site at a position $\boldsymbol{R}$ connected to the origin is in the same cluster as a site a distance ρ along a line perpendicular to $\boldsymbol{R}$ (see figure 2). This can be seen clearly in a mean-field treatment (Redner 1981a, b).

As $p \to p_c(\varphi)$

$$\xi_{\perp}(\varphi) \sim (p - p_c(\varphi))^{-\nu_{\perp}(\varphi)}$$

where $\nu_{\perp}(\varphi) \neq \nu_{\parallel}(\varphi)$ (Domany and Kinzel 1981). It was shown by Klein and Kinzel (1981) that $\nu_{\perp}(\varphi)$ is related to $\nu_{\parallel}(\varphi)$ and the crossover exponent.

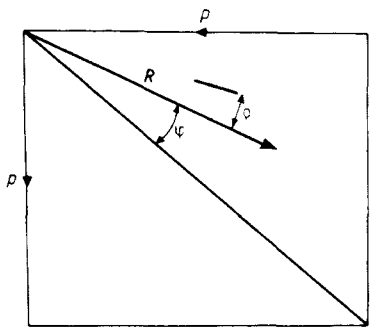

Figure 2. The arrows labelled p are along the diagonal of the face of the square. Percolation is observed along the vector $\boldsymbol{R}$ which makes an angle φ with the diagonal of the face.

In order to gain insight into the directed percolation problem, Domany and Kinzel solved exactly a special case of directed percolation. In their model, vertical bonds are occupied with probability one and horizontal bonds with probability p (see figure 1(*b*)). They obtained the value $\nu_{\parallel} = 2$ for all φ except the special cases $\varphi = \pm\pi/4$.

For $\varphi = \pi/4$ the problem reduces to one-dimensional percolation where $\nu_{\parallel} = 1$ (Reynolds *et al* 1977b). This seems to indicate that $\varphi = \pi/4$ is a special point in the phase diagram of this problem and that there is a crossover from $\nu_{\parallel}(\varphi) = 2$, $\varphi \neq \pi/4$ to $\nu_{\parallel} = 1$ for $\varphi = \pi/4$. At present there exists no treatment of the problem that describes this crossover.

In this comment we derive an exact renormalisation group (RG) which is applicable over the entire range of φ. We find that $\nu_{\parallel} = 1$ for all φ; *there is no crossover*, but the scaling field changes at $\varphi = \pi/4$. Our conclusion is based on an RG that employs the proper nonlinear scaling fields.

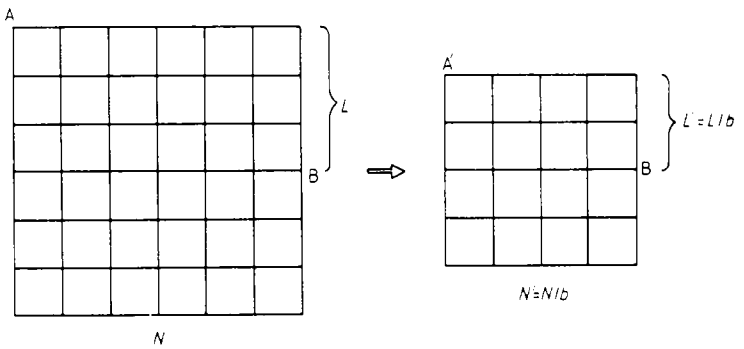

Figure 3. A cell of N sites on a side is mapped to a cell $N' = N/b$ sites on a side. The RG is constructed by requiring the probability that there exists a path from A to B to be equal to the probability that there exists a path from A' to B'.

To construct the RG, consider an $N \times N$ cell on a square lattice (figure 3). We calculate the probability that the site in the upper left-hand corner (labelled A in figure 3) is connected to a site on the right-hand edge of the cell a distance L from the top (labelled B in figure 3). In addition we require that there exists no connected path from A to the right-hand edge which terminates at a point closer to the top of the cell than B. In other words, the path from A to B is the shortest path from A to the right-hand side. (We can also calculate the probability that A is connected to a point on the bottom edge of the cell a distance L from the left-hand side. The resultant transformation is the same.) The probability that such a path exists is given by

$$P_{AB} = C_{NL} p^N q^L \tag{1a}$$

where $C_{NL} = \binom{N+L-1}{N}$ (Domany and Kinzel 1981). Now consider the same problem on a block N/b on a side. The probability that the shortest path from A' (figure 3) to the right-hand edge ending at a point B' which is $L' = L/b$ from the top is given by

$$P_{A'B'} = C_{N'L'} p^{N'} q^{L'} \tag{1b}$$

where $C_{N'L'}$ is identical in form to C_{NL}.

We construct the RG transformation by demanding that these two probabilities be the same, i.e.,

$$\binom{N'+L'-1}{N'} p'^{N'} q'^{L'} = \binom{N+L-1}{N} p^N q^L. \tag{2}$$

(The idea of a cell-to-cell RG transformation was introduced in the pure percolation problem by Reynolds *et al* 1978.)

We do not expect exact results from such a transformation unless the limit $N, L \to \infty$ is taken (Reynolds *et al* 1977a, 1980). In that limit if we define $\omega = L/N = L'/N'$

and use Stirling's approximation we obtain

$$\left[\frac{(1+\omega)^{1+\omega}}{\omega^{\omega}}p'q'^{\omega}\right]^{N/b}=\left[\frac{(1+\omega)^{\omega}}{\omega^{\omega}}pq^{\omega}\right]^{N}. \tag{3}$$

Taking the logarithm of both sides gives

$$\ln\left[\frac{(1+\omega)^{1+\omega}}{\omega^{\omega}}p'q'^{\omega}\right]=b\ln\left[\frac{(1+\omega)^{1+\omega}}{\omega^{\omega}}pq^{\omega}\right]. \tag{4}$$

In this equation $\psi=\ln[(1+\omega)^{1+\omega}pq^{\omega}/\omega^{\omega}]$ is the global nonlinear scaling field.

The RG transformation of equation (4) has two fixed points given by $pq^{\omega}=0$ and $(1+\omega)^{1+\omega}pq^{\omega}/\omega^{\omega}=1$. The former is stable with eigenvalue b^{-1} and the latter unstable with eigenvalue b. The unstable fixed point determines the critical surface, i.e.,

$$(1+\omega)^{1+\omega}pq^{\omega}/\omega^{\omega}=1\Rightarrow p=(1+\omega)^{-1} \tag{5}$$

in agreement with Domany and Kinzel.

Equation (4) is valid for all ω and the eigenvalue $\lambda=b$ implies $\nu_{\|}=1$ for all φ in contrast to the result of Domany and Kinzel. This apparent contradiction can be resolved by examining the nature of the nonlinear scaling field ψ in the neighbourhood of the non-trivial fixed point.

Although it is possible to investigate the non-trivial fixed point for general b and ω, for simplicity we consider the case $b=2$, $\omega=1$. For these values equation (4) becomes

$$\ln(4p'q')=2\ln(4pq) \tag{6}$$

and $p_c=\frac{1}{2}$. This may be written as

$$\ln[1-4(p'-\tfrac{1}{2})^2]=2\ln[1-4(p-\tfrac{1}{2})^2]. \tag{7}$$

In the neighbourhood of $p_c=\frac{1}{2}$, equation (7) reduces to

$$(p'-\tfrac{1}{2})^2=2(p-\tfrac{1}{2})^2. \tag{8}$$

Equation (8), when generalised to arbitrary b and $\omega\neq 0,\infty$ implies that $\nu_{\|}=1$ but that the scaling field is $(p-p_c)^2$ so that

$$\xi_{\|}\sim(p-p_c)^{-2} \tag{9}$$

for $\omega\neq 0$, consistent with the result of Domany and Kinzel.

For $\omega\to 0$

$$(\omega+1)^{\omega+1}pq^{\omega}/\omega^{\omega}\to p \tag{10}$$

so that equation (4) reduces to

$$\ln p'=b\ln p \tag{11}$$

which is the exact RG transformation in $d=1$ (Reynolds *et al* 1977b).

The similarity in the form of equations (4) and (11) would imply that the anisotropic problem with $p_H=1$ and $p_V=p$ is in fact a form of one-dimensional percolation. This is also supported by the fact that $\beta=0$ for both problems (Reynolds *et al* 1978, Klein and Kinzel 1981).

To summarise: in order to have an RG transformation that can treat all values of ω we are forced to conclude that $\nu_{\|}=1$ for all ω where $\nu_{\|}$ is defined by

$$\xi_{\|}\sim x^{-\nu_{\|}} \tag{12}$$

where x is the scaling field. The scaling field, however, in the neighbourhood of the non-trivial fixed point changes from $x = (p - p_c)^2$ for $\omega \neq 0$ to $x = (1-p) = q$ for $\omega = 0$.

It is interesting in the light of the work of Reynolds *et al* (1978, 1980) on the cell-to-cell transformation that the RG of equation (4) cannot be reduced to the form

$$p' = R(p) \tag{13}$$

with $R(p)$ an analytic function in the neighbourhood of the real line for $0 \leqslant p \leqslant 1$.

To illustrate the point we again consider the case $b = 2$, $\omega = 1$ so that the RG reduces to equation (6). Setting $p = 1 - q$ gives

$$p'^2 - p' + 4p^2(1-p)^2 = 0. \tag{14}$$

Therefore

$$p' = \tfrac{1}{2} \pm \tfrac{1}{2}\{1 - [4p(1-p)]^2\}^{1/2}. \tag{15}$$

Equation (15) has a non-analyticity at $p = \frac{1}{2}$ which is the non-trivial fixed point. At $p = \frac{1}{2}$, $dp'/dp = \pm\infty$.

This is of course a consequence of the fact that the scaling field in the neighbourhood of $p = p_c$ is $(p - p_c)^2$ and not $p - p_c$.

In conclusion, we have demonstrated that the use of proper scaling fields eliminates the crossover as $\omega \to 0$. In these proper variables the critical properties become those of one-dimensional percolation for all ω.

We have also shown by providing a counter example that the cell-to-cell transformation introduced by Reynolds *et al* cannot in all cases be reduced to the form $p' = R(p)$ where $R(p)$ is an analytic function of p in the neighbourhood of the real p axis for $0 \leqslant p \leqslant 1$.

Acknowledgment

I would like to thank S Redner and H E Stanley for useful comments on the manuscript.

References

Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Domany E and Kinzel W 1981 *Phys. Rev. Lett.* **47** 5
Essam J W 1980 *Rep. Prog. Phys.* **43** 833
Grassberger P and de la Torre A 1979 *Ann. Phys., NY* **122** 373
Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen.* **14** L163
Klein W and Kinzel W 1981 *J. Phys. A: Math. Gen.* **14** L405
Redner S 1981a *Preprint*
—— 1981b *Preprint*
Redner S and Brown A C 1981 *J. Phys. A: Math. Gen.* **14** L285
Reynolds P J 1981 *Preprint*
Reynolds P J, Klein W and Stanley H E 1977a *J. Phys. C: Solid State Phys.* **10** L167
Reynolds P J, Stanley H E and Klein W 1977b *J. Phys. A: Math. Gen.* **10** L203
—— 1978 *J. Phys. A: Math. Gen.* **11** L199
—— 1980 *Phys. Rev.* B **21** 1223
Schlögl F 1972 *Z. Phys.* **253** 147
Stauffer D 1979 *Phys. Rep.* **54** 1